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THE BANACH-TARSKI-PARADOX

To choose one sock from each of infinitely many pairs of socks requires the Axiom of Choice, but for shoes the Axiom is not needed.

(Bertrand Russell)

The *Axiom of Choice* was first formulated by *Ernst Zermelo*[1] in 1904, and since that time, there is a controverse discussion about it till nowadays. The reason for this is the fact that it is possible to derive some properties which seem to be contrary to our experience. One of the most popular paradoxes of this kind is the *Banach-Tarski-Paradox.*

1 Basic Facts

AXIOM OF CHOICE Suppose a class of non-empty sets $\{M_i \mid i \in I\}$ with non-empty index set I. Then there exists a "choice-mapping"

$$f : I \to \bigcup_{i \in I} M_i$$

with $f(i) \in M_i$ for all $i \in I$.

1.1 Definition We define the *closed unit ball*

$$S_3 \;:=\; \{x \in \mathbb{R}^3 \mid \|x\| \leq 1\}$$

and the unit sphere

$$\partial S_3 \;:=\; \{x \in \mathbb{R}^3 \mid \|x\| = 1\}.$$

1.2 Definition SO_3 is defined to be the group of all orthogonal 3×3-matrices with determinant 1. If $A \in SO_3$ is arbitrary, then it is always possible to choose an orthonormal base of $\mathbb{R}^3$ in such a way that A can be brought to the form

$$\begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

1.3 Definition Let G be a group and $X \neq \emptyset$ an arbitrary set. G *operates on* X, if there exists a mapping $\bullet : G \times X \to X,\ (g, x) \mapsto g \bullet x$ with
(O_1) $(gh) \bullet x \;=\; g \bullet (h \bullet x)$ for all $g, h \in G,\ x \in X$;
(O_2) $e \bullet x \;=\; x$ for the neutral element e and all $x \in X$.

1.4 Definition Suppose an arbitrary set X and a group G which operates on X. A subset $E \subseteq X$ is called *G-paradox*, if there exist some numbers $n, m \in \mathbb{N}$, pairwise disjoint sets $A_1, ..., A_n, B_1, ..., B_m \subseteq E$ and elements $g_1, ..., g_n, h_1, ..., h_m \in G$ such that

$$E \;\equiv\; \bigcup_{i=1}^{n} g_i \bullet A_i \;\equiv\; \bigcup_{j=1}^{m} h_j \bullet B_j.$$

[1] Ernst Friedrich Ferdinand Zermelo, born on 27.07.1871 in Berlin, and died on 21.05.1953 in Freiburg.

To illustrate the term "paradox" in this definition we can look at the following

1.5 Example Suppose $E := X := S_3$ to be the set of all points of the closed unit ball and $G := SO_3$ the group of rotations in $\mathbb{R}^3$, which should operate on X. Suppose E to be G-paradox, then we could split E in pairwise disjoint parts $A_1, ..., A_n, B_1, ..., B_m \subseteq E$ and apply some rotations $g_1, ..., g_n, h_1, ..., h_m \in G$ to them. Afterwards we put together the rotated parts and obtain ***two closed unit balls*** $E_1 = \bigcup_{i=1}^n \; g_i \bullet A_i$ and $E_2 = \bigcup_{j=1}^m \; h_j \bullet B_j$ identical with E! So by splitting the ball and rotating its parts in some way we would double the volume! Indeed paradox![2]

Suppose a group F which is restricted only by the group axioms and nothing else. We can call such an "ideal" group a ***free group***. Such a free group is usually constructed in the following way:

1.6 Definition Suppose a set of ***letters*** $\{f_i \mid i \in I\}$ and define some new symbols $\{f_i^{-1} \mid i \in I\}$. Now we build ***words*** by putting these letters together like $f_{i_1}^{\epsilon_1} \cdots f_{i_r}^{\epsilon_r}$ with $\epsilon_1, ..., \epsilon_r \in \{-1, 1\}$. We call such a word a ***reduced word*** if we put out all terms of the form $f_i^{\epsilon} f_i^{-\epsilon}$. Furthermore we define the ***empty word*** e.
If F is now defined to be the set of all reduced words over $\{f_i, f_i^{-1} \mid i \in I\}$ together with the empty word e, then F forms a group, called the ***free group*** of ***rank*** $|I|$ over the ***free generators*** $\{f_i \mid i \in I\}$, where the inverse of a reduced word $f := f_{i_1}^{\epsilon_1} \cdots f_{i_r}^{\epsilon_r}$ is given by $f^{-1} := f_{i_r}^{-\epsilon_r} \cdots f_{i_1}^{-\epsilon_1}$.

Before we start regarding the Banach-Tarski-Paradox, we will first observe (as some kind of "warm-up")...

2 A Nonmeasurable Set of Real Numbers

Let $\mu(X)$ denote the Lebesgue measure of some set $X \subseteq \mathbb{R}$. μ is countably additive, and for $\mu([a, b])$ we obtain $\mu([a, b]) = b - a$.
Now we define the relation $\underset{R}{\sim}$ by

$$x \underset{R}{\sim} y \;:\Longleftrightarrow\; x - y \in \mathbb{Q}.$$

It is easily shown that $\underset{R}{\sim}$ is an equivalence relation, so we can partition $X := [0, 1]$ into the pairwise disjoint equivalence classes $[x] := \{y \in X \mid y \underset{R}{\sim} x\}$; we denote $\mathcal{M} := \{[x] \mid x \in X\}$.
Using the ***Axiom of Choice*** we find a set of representatives $M \subseteq X$ for $\mathcal{M}$, so for every $x \in \mathbb{R}$ there exists an unique $y \in M$ and an unique $q \in \mathbb{Q}$ such that $x = y + q$.
If we now define

$$M_q \;:=\; \{y + q \mid y \in M\}$$

for each $q \in \mathbb{Q} \cap [-1, 1]$, then all M_qs are disjoint and $\{M_q \cap [0, 1]\}$ form a countable partition of $[0, 1]$.
Suppose that M is measurable, then $\mu(M)$ is either zero or positive and $\mu(M_q) = \mu(M)$ for all $q \in \mathbb{Q} \cap [-1, 1]$.

[2] "At first glance, the Banach-Tarski Decomposition seems to contradict some of our intuition about physics – e.g., the Law of Conservation of Mass, from classical Newtonian physics. Consequently, the Decomposition is often called the Banach-Tarski Paradox. But actually, it only yields a complication, not a contradiction. If we assume a uniform density, only a set with a defined volume can have a defined mass. The notion of "volume" can be defined for many subsets of $\mathbb{R}^3$, and beginners might expect the notion to apply to all subsets of $\mathbb{R}^3$, but it does not. More precisely, Lebesgue measure is defined on some subsets of $\mathbb{R}^3$, but it cannot be extended to all subsets of $\mathbb{R}^3$ in a fashion that preserves two of its most important properties: the measure of the union of two disjoint sets is the sum of their measures, and measure is unchanged under translation and rotation. Thus, the Banach-Tarski Paradox does not violate the Law of Conservation of Mass; it merely tells us that the notion of "volume" is more complicated than we might have expected." (Eric Schechter)

If $\mu(M) = 0$, then $\mu([0,1]) \leq \sum_q \mu(M_q) = 0$, which is obviously false.
But if on the other hand $\mu(M_q) > 0$, then

$$\mu([-1,2]) \; \geq \sum_{q \in \mathbb{Q} \cap [-1,1]} \mu(M_q) \; = \; \infty,$$

which is also impossible!
So M is not measureable! #

3 The Banach-Tarski-Paradox

We know want to realize the idea descripted in Example 1.5.
Remembering the definitions 1.6 and 1.4 we can show the following

3.1 Lemma Suppose F to be a free group of rank 2 which operates on itself by multiplication from the left: $f \bullet \tilde{f} := f\tilde{f}$. Then F is F-paradox.

Proof: Suppose $F := \langle \{\sigma, \tau\} \rangle$ and $W(\rho)$ to be the set of all words in F which start with an ρ on the left. Then we get the following partition of F:

$$F \;=\; \{e\} \cup W(\sigma) \cup W(\sigma^{-1}) \cup W(\tau) \cup W(\tau^{-1}),$$

and since $\sigma \bullet W(\sigma^{-1}) = \{e\} \cup W(\sigma^{-1}) \cup W(\tau) \cup W(\tau^{-1})$ and $\tau \bullet W(\tau^{-1}) = \{e\} \cup W(\tau^{-1}) \cup W(\sigma) \cup W(\sigma^{-1})$, we obtain

$$F = W(\sigma) \cup \sigma \bullet W(\sigma^{-1}) \qquad \text{and} \qquad F = W(\tau) \cup \tau \bullet W(\tau^{-1}),$$

so F is F-paradox. #

3.2 Lemma Suppose a group G which operates on the set X. Furthermore let G be G-paradox and suppose that every $g \in G \setminus \{e\}$ has no fixpoint.
Then X is also G-paradox. Especially we have by using Lemma 3.1:
If F is a free group of rank 2 without non-trivial fixpoints, then X is F-paradox.

Proof: Suppose the notations like in Definition 1.4. We again define an equivalence relation $\underset{R}{\sim}$ by

$$x \underset{R}{\sim} y \; :\Longleftrightarrow \; \exists_{g \in G} \; x = g \bullet y.$$

If again $\mathcal{M}$ is the set of all equivalence classes, then we can use the *Axiom of Choice* to find a set $M \subseteq X$ of representatives for $\mathcal{M}$.
Then we can show X can be written as the disjoint union

$$X = \bigcup_{g \in G} g \bullet M.$$

If we now define the sets

$$A_i^* \; := \bigcup_{g \in A_i} g \bullet M \qquad \text{and} \qquad B_j^* \; := \bigcup_{g \in B_j} g \bullet M \text{ for all } i, j,$$

then these new sets are also pairwise disjoint and we find

$$X \;=\; \bigcup_{i=1}^{n} g_i \bullet A_i^* \;\overset{!!}{=}\; \bigcup_{j=1}^{m} h_j \bullet B_j^*,$$

so X is also G-paradox. #

3.3 Lemma If we define $\phi, \psi \in SO_3$ by

$$\phi := \begin{pmatrix} \frac{1}{3} & -\frac{2\sqrt{2}}{3} & 0 \\ \frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \psi := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{2\sqrt{2}}{3} \\ 0 & \frac{2\sqrt{2}}{3} & \frac{1}{3} \end{pmatrix},$$

then $\langle \phi, \psi \rangle$ is free of rank 2.

Proof: (Sketch) Finally we have to show that an reduced form $w \in \langle \phi, \psi \rangle$ equals e if and only if this reduced form is empty.
Therefore it is enough to show (by induction) that for every element w of length n, which ends on the right side with an ϕ, there exist numbers $k, l, m \in \mathbb{Z}$, $3 \not| \, l$, such that

$$w \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 3^{-n} \begin{pmatrix} k \\ l\sqrt{2} \\ m \end{pmatrix} \neq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

so w can't be the identity. #

3.4 Theorem (HAUSDORFF-PARADOX) There exists a countable set $D \subseteq \partial S_3$ such that $\partial S_3 \setminus D$ is SO_3-paradox.

Proof: By Lemma 3.3 we know that there exists a free (sub)group F in SO_3 of rank 2, generated by two rotations of SO_3. Since a non-trivial rotation in SO_3 leaves exactly two(!) points of X invariant (the points where the rotation axis breaks through the sphere), we can define the set D in ∂S_3 of all points which are invariant under at least one rotation in F, so D is countable, since F is countable.
Furthermore, F operates on $\partial S_3 \setminus D$ without non-trivial fixpoint, so we can use Lemma 3.2 to show that $\partial S_3 \setminus D$ is F-paradox, and therefore also SO_3-paradox. #

3.5 Definition Let the group G operate on X.
We call $R, S \subseteq X$ ***congruent*** if there exists a $g \in G$ such that $S = g \bullet R$.
We call $A, B \subseteq X$ ***equally divisible under*** G if we can split A resp. B into the same number of disjoint subsets $A_1, ..., A_n$ resp. $B_1, ..., B_n$ such that A_i is congruent to B_i for all $i = 1, ..., n$.

3.6 Lemma If $D \subseteq \partial S_3$ is countable, then ∂S_3 and $\partial S_3 \setminus D$ are equally divisible under G (in two parts).

Proof: (Sketch) In a first step one has to show that there exists an angle ψ and a corresponding rotation matrix R_ψ such that the sets D, $R_\psi \cdot D$, $R_\psi^2 \cdot D$, ... are pairwise disjoint. The trick: 'Counting' the angles, where the sets are not disjoint, gives countably many such angles, so there are uncountably many angles for our purpose left!
Now we can split

$$\partial S_3 = \left(\bigcup_{n=0}^{\infty} R_\psi^n \cdot D \right) \cup \left(\partial S_3 \setminus \bigcup_{n=0}^{\infty} R_\psi^n \right) =: A_1 \cup A_2$$

and

$$\partial S_3 \setminus D = \left(\bigcup_{n=1}^{\infty} R_\psi^n \cdot D \right) \cup \left(\partial S_3 \setminus \bigcup_{n=0}^{\infty} R_\psi^n \right) =: B_1 \cup B_2.$$

Since $B_1 = R_\psi \cdot A_1$ and $B_2 = A_2$, the parts are congruent. #

Now we can apply this Lemma on Theorem 3.4 to show the following

3.7 Theorem ∂S_3 is SO_3-paradox.

Proof: By 3.4, we can write $\partial S_3 \setminus D$ as the disjoint unions

$$\partial S_3 \setminus D \;=\; \bigcup_{i=1}^{n} g_i \bullet A_i \;=\; \bigcup_{j=1}^{m} h_j \bullet B_j$$

with pairwise disjoint sets $A_1, ..., A_n, B_1, ..., B_m \subseteq \partial S_3 \setminus D$.
On the other hand, we obtain from 3.6, that

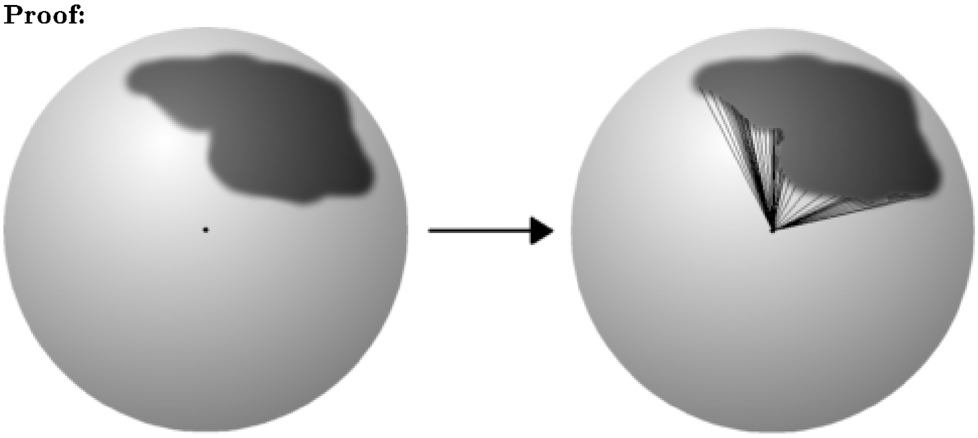
$$\partial S_3 \setminus D \;=\; C_1 \cup C_2 \qquad \text{and} \qquad \partial S_3 \;=\; f_1 \bullet C_1 \cup f_2 \bullet C_2$$

with disjoint sets C_1, C_2.
We can complete the proof by observing

$$A^*_{i,k} \;:=\; A_i \cap g_i^{-1} \bullet C_k, \quad B^*_{j,k} \;:=\; B_j \cap h_j^{-1} \bullet C_k, \quad g^*_{i,k} \;:=\; f_k g_i \quad \text{and} \quad h^*_{j,k} := f_k h_j.$$ #

3.8 Corollary (Banach-Tarski-Paradox) S_3 is SO_3-paradox.

Proof:



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Literature

Chapter 2 is based on the book

The Axiom of Choice, *by Thomas J. Jech*, North-Holland Publishing Company, 1973

the other parts follow the descripions in the script Masstheorie by *Carsten Schütt.*